On Frequency Weighting for the H_∞ and H_2 Control Design of Flexible Structures

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Introduction

The H_∞ and the related H_2 controller design methodologies allow for the design of control systems that meet tracking requirements and, at the same time, maintain the binding disturbance rejection properties. In order to achieve this, the problem should be appropriately defined in the quantitative terms. For example, the frequency shaping filters are used to define tracking requirements, or disturbance rejection performance of the closed-loop system. These frequency dependent weights filters are used only in the controller design stage. They add to the complexity of the problem, since in the process of design the number of system equations varies and their parameters are modified. As it is stated by Voth *et al.* In Ref.1, p.55, "The selection of the controller gains and filters as well as the controller architecture is an iterative, and often tedious, process which relies heavily on the designers' experience." It is shown in this paper that this task is simplified in the case of flexible structure control. If a structure model is in the modal representation, then the addition of a filter is equivalent to the multiplication of each row of the plant input matrix by a constant. The ith constant is the filter gain at the ith natural frequency of the structure. In this way each natural mode is weighted separately. This approach addresses the system performance at the mode level, which simplifies otherwise may be *ad hoc* and tedious process.

Properties of Flexible Structures and Filters

We assume that a flexible structure is in the modal representation. Its transfer function G has the state space representation (A,B,C), with n degrees of freedom (or number of flexible structure modes), $2n$ states, p inputs, and q outputs. Denote ω_i the ith natural frequency, and ζ_i the ith modal damping, $i=1,...,n$. We assume low damping ($\zeta_i < 0.1$ for all modes), and distinct natural frequencies. In the modal representation the system matrix A is block diagonal with 2×2 blocks on the diagonal, and B, C are divided into $2 \times p$ and $q \times 2$ blocks, see Ref.2, pp.12-14

$$A = diag(A_i), \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \quad A_i = \begin{bmatrix} -\zeta_i \omega_i & \omega_i \\ -\omega_i & -\zeta_i \omega_i \end{bmatrix} \tag{1}$$

where $i=1,...,n$. Denote the transfer function of the ith mode as $G_i = C_i(j\omega I - A_i)^{-1} B_i$, then, one obtains the following decomposition of transfer function in modal coordinates

$$\text{a) } G(\omega) = \sum_{i=1}^{n} G_i(\omega) \qquad \text{b) } G(\omega_i) \cong G_i(\omega_i),\ i=1,...,n \tag{2}$$

Part (a) can be derived by introducing A, B, and C as in Eq.(1) to the definition of the transfer function. Part (b) follows from Part (a), by noting that the response at frequency ω_i is dominated by the ith mode, that is, $\left\|G_j(\omega_i)\right\|_2 << \left\|G_i(\omega_i)\right\|_2$ for $i \neq j$.

Consider a filter, with a diagonal transfer function $F(\omega)$. The diagonal input (output) filter represents lack of coupling between the inputs (or outputs). Denote α_i the magnitude of the filter response at the ith natural frequency, $\alpha_i = \left|F(\omega_i)\right|$. Filter is smooth if the slope of its transfer function near the structural resonance is small when compared to the slope of the structure near the resonance. With the above assumptions the following property of the H_∞ norm of a structure with a filter is valid

$$\text{a) } \left\|G\right\|_\infty \cong \max_{i\in[1,n]}(\left\|G_i\right\|_\infty) \quad \text{and} \quad \text{b) } \left\|GF\right\|_\infty \cong \max_{i\in[1,n]}(\left\|G_i \alpha_i\right\|_\infty) \tag{3}$$

In order to prove part (a), note that

$$\left\|G\right\|_\infty = \sup_\omega \bar{\sigma}(G(\omega)) \cong \max_{i\in[1,n]} \bar{\sigma}(G(\omega_i)) \cong \max_{i\in[1,n]} \bar{\sigma}(G_i(\omega_i)) \tag{4a}$$

Both approximations in the above equation hold because resonance response of the ith mode dominates the structural response, as given in Eq.(2). However, since $\|G_i\|_\infty \cong \bar{\sigma}(G_i(\omega_i))$, therefore (3), part (a) is valid. In order to prove (b) note that for the smooth F the transfer function GF preserves the properties of a flexible structure given by Eq.(2), thus

$$\|GF\|_\infty = \sup_{\omega} \bar{\sigma}(G(\omega)F(\omega)) \cong \max_{i\in[1,n]} \bar{\sigma}(G(\omega_i)F(\omega_i)) \cong \max_{i\in[1,n]} \bar{\sigma}(G_i(\omega_i)\alpha_i) \tag{4b}$$

In the above approximation we took into consideration the fact that $\sigma_k(GF) = \sigma_k(G|F|)$, which can be proven as follows

$$\sigma_k^2(GF) = \lambda_k(F^*G^*GF) = \lambda_k(FF^*G^*G) = \lambda_k(|F|^2 G^*G) = \lambda_k(|F|G^*G|F|) = \sigma_k^2(G|F|) \tag{4c}$$

Eq.(3) says that the largest modal peak response of a lightly damped structure determines the worst case response. Similar property holds for the 2-norm of a structure with a filter

$$\text{a) } \|G\|_2^2 \cong \sum_{i=1}^{n} \|G_i\|_2^2 \quad \text{and} \quad \text{b) } \|GF\|_2^2 \cong \sum_{i=1}^{n} \|G_i\alpha_i\|_2^2 \tag{5}$$

Eq.(5) says that the rms response of a lightly damped structure is approximately an rms sum of responses of each mode. Also, a norm of a smooth filter in series with a flexible structure is approximately equal to the norm of a structure scaled by the filter gains at natural frequencies. Note that similar result to Eq.(5) holds for a structure with a filter at the output.

Approximate Frequency Weighting

In controller design the inputs of the plant transfer function are separated into two groups: the exogenous input w (that includes commands and disturbances), and the actuator input u. The system outputs consists of the performance output z, at which performance is evaluated, and the sensed output y. The H_∞ (H_2) control problem consists of determining the stabilizing controller transfer function K such that the H_∞ (H_2) norm of the closed-loop transfer function G, from w to z, is minimized over all realizable controllers K.

Frequency weighting of the exogenous inputs and performance outputs is a standard approach in the H_∞ (H_2) design to define the required closed-loop properties, (see for example Ref.3). In this case a plant is augmented with the input and/or output shaping filters, forming an augmented plant model. Consider, for example, input shaping. Denote F the transfer function of the input filter, and assume that it is smooth. The transfer function from w to z with the input filter is GF. The inputs of G are shaped independently, therefore the filter transfer function matrix is square and diagonal.

Introduce the transfer function $\hat{G} = \sum_{i=1}^{n} \hat{G}_i$ where $\hat{G}_i = C_i(j\omega I - A_i)^{-1}\hat{B}_i$ and $\hat{B}_i = B_i\alpha_i$.

Above, $\hat{G}_i$ is a transfer function G_i with the scaled input matrix B_i. We will show that the H_∞ norms of both transfer function are approximately equal.

$$\|GF\|_\infty \cong \|\hat{G}\|_\infty \tag{6}$$

One can prove it using Eq.(3b), obtaining $\|GF\|_\infty \cong \max_{i\in[1,n]}\|G_i\alpha_i\|_\infty = \max_{i\in[1,n]}\|\hat{G}_i\|_\infty \cong \|\hat{G}\|_\infty$.

Equation (6) shows that the application of the input filter for the H_∞ performance modeling is equivalent to the scaling of the $2\times p$ input matrix B_i with α_i, where α_i is the magnitude of the filter transfer function at the resonant frequency ω_i. For the H_∞ output filter one obtains

$$\|FG\|_\infty \cong \|\overline{G}\|_\infty \tag{7}$$

where $\overline{G} = \sum_{i=1}^{n} \overline{G}_i$, $\quad \overline{G}_i = \overline{C}_i(j\omega I - A_i)^{-1}B_i$, and $\quad \overline{C}_i \doteq \alpha_i C_i$.

For the H_2 controller the 2-norm of the transfer function GF is used as a system performance measure. We assume that the flexible structure is represented in the modal coordinates, and for the input filter we obtain.

$$\|GF\|_2 \cong \|\hat{G}\|_2 \tag{8}$$

We prove it using Eq.(5) obtaining $\|GF\|_2^2 \cong \sum_{i=1}^{n} \|G_i \alpha_i\|_2^2 = \sum_{i=1}^{n} \|\hat{G}_i\|_2^2 = \|\hat{G}\|_2^2$.

Equation (8) shows that the application of the input filter for the H_2 performance modeling is equivalent to the scaling of the $2 \times p$ modal input matrix B_i with α_i. For the H_2 output filter one obtains

$$\|FG\|_2 \cong \|\overline{G}\|_2. \tag{9}$$

Note that Eqs.(6)-(9) preserve the order and the physical interpretation of the transfer function, and the corresponding state variables. This simplifies the controller design process, since the relationship between the filter gains and the system performance is readily available.

Example

Consider a steel truss as in Fig.1. For this truss $l_1 = 10$ cm, $l_2 = 8$ cm, cross section area is 1 cm^2. The disturbance w is applied at node 7 in z direction, the performance z is measured at node 21, z direction; the input u is applied at node 20 in z direction, and the output y is a displacement of node 28, z direction. The open-loop transfer function from the disturbance to the performance is shown in Fig.2, solid line. The disturbance input is filtered with a low-pass filter, $F(s) = 1/(1 + 0.011s)$, the magnitude of its transfer function is shown in the same figure, dot-dashed line. The resulting transfer function of the structure and the filter is represented by the dotted line. The equivalent structure with filter was obtained by scaling the disturbance input, according to Eq.(6), and the magnitude of its transfer function is shown in Fig.2, dashed line. It is clear from that figure that the structure with the filter, and the structure with the scaled disturbance input have very similar frequency characteristics, and their norms are $\|G\|_\infty = 2.6903$, $\|G\|_2 = 453.2945$ for the structure with the filter, and $\|G\|_\infty = 2.6911$, $\|G\|_2 = 453.5661$ for the structure with the scaled disturbance input.

Two frequency weighted H_∞ controllers for this structure were designed. The first one is based on the structure with a filter, while the second is based on the structure with scaled input matrix. The open- closed-loop transfer functions are shown in Fig.3. The closed loop performance of the structure with the filter, and with the scaled input is almost identical. The closed-loop H_∞ norms are as follows: $\|G_{cl}\|_\infty = 0.09681$ for the structure with the filter, and $\|G_{cl}\|_\infty = 0.09676$ for the structure with the scaled disturbance input. In a similar manner H_2 controllers were designed. The closed-loop H_2 norms are as follows: $\|G_{cl}\|_2 = 108.6295$ for the structure with the filter, and $\|G_{cl}\|_2 = 108.7181$ for the structure with the scaled disturbance input.

Conclusions

It has been shown that for flexible structures the frequency shaping of the system properties with input (output) filters is equivalent to the scaling the modal input (output) matrix of the plant. This approach simplifies controller design process. Instead of introducing new state variables, one modifies the gains of the modal input matrix. This is possible since the modal states related to the gains are weakly coupled, such that the modification of one state (or one gain) weakly influence the others. In addition, physical interpretation of the states remains unchanged and is related to the corresponding gains.

References

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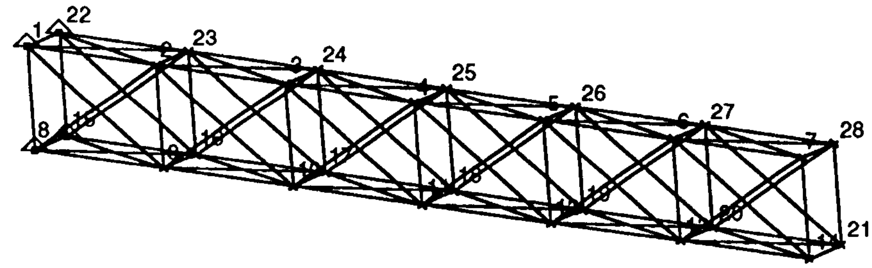

Figure 1. Truss

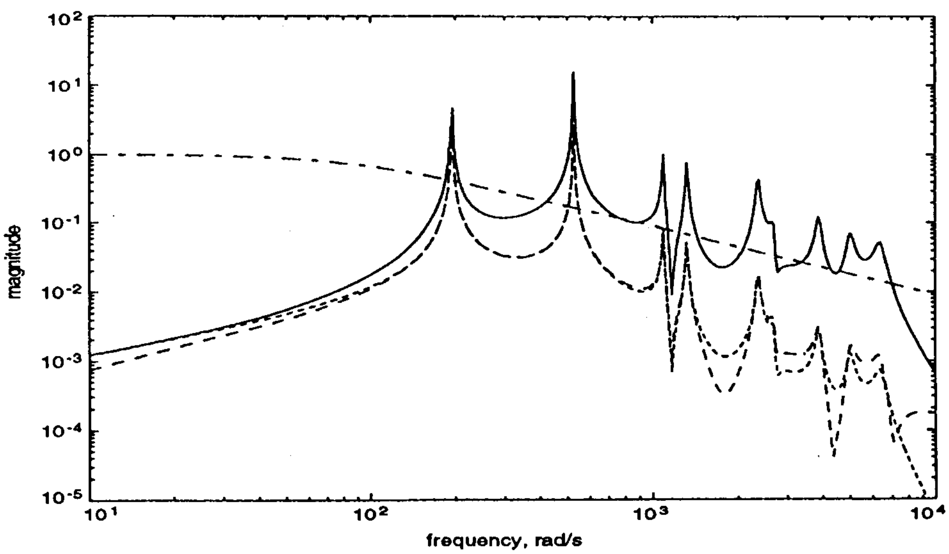

Figure 2. Magnitudes of the open-loop transfer function of the truss (solid line), filter (dash-dot line), truss with filter (dotted line), and truss with scaled disturbance input (dashed line).